Non-minimal curvature-matter couplings in modified gravity

Orfeu Bertolami* and Jorge Páramos†

Instituto Superior Técnico
Departamento de Física and Instituto de Plasmas e Fusão Nuclear,
Av. Rovisco Pais 1, 1049-001 Lisboa, Portugal

Tiberiu Harko‡

Department of Physics and Center for Theoretical and Computational Physics,
The University of Hong Kong, Pok Fu Lam Road, Hong Kong

Francisco S. N. Lobo§

Institute of Cosmology & Gravitation, University of Portsmouth, Portsmouth PO1 2EG, UK and
Centro de Astronomia e Astrofísica da Universidade de Lisboa,
Campo Grande, Ed. C8 1749-016 Lisboa, Portugal

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Recently, in the context of $f(R)$ modified theories of gravity, it was shown that a curvature-matter coupling induces a non-vanishing covariant derivative of the energy-momentum, implying non-geodesic motion and, under appropriate conditions, leading to the appearance of an extra force. We study the implications of this proposal and discuss some directions for future research.

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I. INTRODUCTION

Current experimental evidence indicates that gravitational physics is in agreement with Einstein's theory of General Relativity (GR) to considerable accuracy (for thorough discussions see [1]); however, quite fundamental questions suggest that it is unlikely that GR stands as the ultimate description of gravity. Actually, difficulties arise from various corners, most particularly in connection to the strong gravitational field regime and the existence of spacetime singularities. Quantization is a possible way to circumvent these problems, nevertheless, despite the success of gauge field theories in describing the electromagnetic, weak, and strong interactions, the description of gravity at the quantum level is still missing, despite outstanding progress achieved, for instance, in the context of superstring/M-theory.

Furthermore, in fundamental theories that attempt to include gravity, new long-range forces often arise in addition to the Newtonian inverse-square law. Even if one assumes the validity of the Equivalence Principle, Einstein's theory does not provide the most general way to establish the spacetime metric. There are also important reasons to consider additional fields, especially scalar fields. Although the latter appear in unification theories, their inclusion predicts a non-Einsteinian behaviour of gravitating systems. These deviations from GR include violations of the Equivalence Principle, modification of large-scale gravitational phenomena, and variation of the fundamental couplings.

On large scales, recent cosmological observations lead one to conclude that our understanding of the origin and evolution of the Universe based on GR requires that most of the energy content of the Universe is in the form of currently unknown dark matter and dark energy components that may permeate much, if not all spacetime. Indeed, recent Cosmic Microwave Background Radiation (CMBR) data indicate that our Universe is well described, within the framework of GR, by a nearly flat Robertson-Walker metric. Moreover, combination of CMBR, supernovae, baryon acoustic oscillation and large scale structure data are consistent with each other only if, in the cosmic budget of energy, dark energy corresponds to about 73% of the critical density, while dark matter to about 23% and baryonic matter to only about 4%. Several models have been suggested to address issues related to these new dark states. For dark energy, one usually considers the so-called "quintessence" models, which involves the slow-roll down of a scalar field along a smooth potential, thus inducing the observed accelerated expansion (see [2] for a review). For dark matter, several weak-interacting particles (WIMPs) have been suggested, many arising from extensions to the Standard Model (*e.g.* axions, neutralinos). A scalar field can also account for an unified model of dark energy and dark matter [3]. Alternatively, one can implement this unification through an exotic equation of state, such as the generalized Chaplygin gas [4].

However, recently a different approach has attracted some attention, namely the one where one considers a generalization of the action functional. The most straight forward approach consists in replacing the linear scalar curvature term in the Einstein-Hilbert action by a func-

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*Electronic address: orfeu@cosmos.ist.utl.pt
†Electronic address: jorge.paramos@ist.utl.pt
‡Electronic address: harko@hkucc.hku.hk
§Electronic address: francisco.lobo@port.ac.uk

tion of the scalar curvature, $f(R)$. In this context, a renaissance of $f(R)$ modified theories of gravity has recently been verified in an attempt to explain the late-time accelerated expansion of the Universe (see for instance Refs. [5, 6] for recent reviews). One could alternatively, resort to other scalar invariants of the theory and necessarily analyze the observational signatures and the parameterized post-Newtonian (PPN) metric coefficients arising from these extensions of GR. In the context of dark matter, the possibility that the galactic dynamics of massive test particles may be understood without the need for dark matter was also considered in the framework of $f(R)$ gravity models [7]. Despite the extensive literature on these $f(R)$ models, an interesting possibility has passed unnoticed till quite recently. It includes not only a non-minimal scalar curvature term in the Einstein-Hilbert Lagrangian density, but also a non-minimal coupling between the scalar curvature and the matter Lagrangian density [8] (see also Ref. [9] for related discussions). It is interesting to note that nonlinear couplings of matter with gravity were analyzed in the context of the accelerated expansion of the Universe [10], and in the study of the cosmological constant problem [11]. In this contribution we discuss various aspects of this proposal.

This work is organized as follows: in the following Section, the main features of this novel model are presented. In Section III, the issue of the degeneracy of Lagrangian densities, actually a feature well known in GR [12–14], is addressed in the context of the new non-minimally coupled model [15]. In Section IV, the scalar-tensor representation of the model is presented, with particular emphasis on the new features and difficulties encountered in the new model. These issues are quite relevant, as they allow one to properly obtain the PPN parameters β and γ and show that they are consistent with the observations [16]. Section V, addresses the compatibility of the model with the astrophysical condition for stellar equilibrium [17]. In Section VI, a further generalization of the model is discussed and an upper bound on the extra acceleration introduced by the new non-minimal coupling is obtained [18]. Finally, in Section VI our conclusions are presented and objectives for further research are discussed.

Throughout this work, the convention $8\pi G = 1$ and the metric signature $(-, +, +, +)$ are used.

II. LINEAR CURVATURE-MATTER COUPLINGS

The action for curvature-matter couplings, in $f(R)$ modified theories of gravity [8], takes the following form

$$S = \int \left[\frac{1}{2} f_1(R) + [1 + \lambda f_2(R)] \mathcal{L}_m \right] \sqrt{-g} \, d^4x, \quad (1)$$

where $f_i(R)$ (with $i = 1, 2$) are arbitrary functions of the curvature scalar R and $\mathcal{L}_m$ is the Lagrangian density corresponding to matter and λ is a constant. Since the

matter Lagrangian is not modified in the total action, these may be called modified gravity models with a non-minimal coupling between matter and geometry.

Varying the action with respect to the metric $g_{\mu\nu}$ yields the field equations, given by

$$F_1 R_{\mu\nu} - \frac{1}{2} f_1 g_{\mu\nu} - \nabla_\mu \nabla_\nu F_1 + g_{\mu\nu} \Box F_1 = (1 + \lambda f_2) T_{\mu\nu}$$
$$-2\lambda F_2 \mathcal{L}_m R_{\mu\nu} + 2\lambda (\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box) \mathcal{L}_m F_2 \,, \quad (2)$$

where one denotes $F_i(R) = f_i'(R)$, and the prime denotes differentiation with respect to the scalar curvature. The matter energy-momentum tensor is defined as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta(g^{\mu\nu})}. \quad (3)$$

Now, taking into account the generalized Bianchi identities, one deduces the following generalized covariant conservation equation

$$\nabla^\mu T_{\mu\nu} = \frac{\lambda F_2}{1 + \lambda f_2} [g_{\mu\nu} \mathcal{L}_m - T_{\mu\nu}] \nabla^\mu R. \quad (4)$$

It is clear that the non-minimal coupling between curvature and matter yields a non-trivial exchange of energy and momentum between the geometry and matter fields [16].

Considering, for instance, the energy-momentum tensor for a perfect fluid,

$$T_{\mu\nu} = (\rho + p) U_\mu U_\nu + p g_{\mu\nu} \,, \quad (5)$$

where ρ is the energy density and p is the pressure, respectively. The four-velocity, U_μ, satisfies the conditions $U_\mu U^\mu = -1$ and $U^\mu U_{\mu;\nu} = 0$. Introducing the projection operator $h_{\mu\nu} = g_{\mu\nu} + U_\mu U_\nu$, one can show that the motion is non-geodesic, and governed by the following equation of motion for a fluid element

$$\frac{dU^\mu}{ds} + \Gamma^\mu_{\alpha\beta} U^\alpha U^\beta = f^\mu \,, \quad (6)$$

where the extra force, f^μ, appears and is given by

$$f^\mu = \frac{1}{\rho + p} \left[\frac{\lambda F_2}{1 + \lambda f_2} (\mathcal{L}_m - p) \nabla_\nu R + \nabla_\nu p \right] h^{\mu\nu}. \quad (7)$$

One verifies that the first term vanishes for the specific choice of $\mathcal{L}_m = p$, as noted in [19]. However, as pointed out in [15], this is not the unique choice for the Lagrangian density of a perfect fluid, as will be outlined below.

III. PERFECT FLUID LAGRANGIAN DESCRIPTION

The novel coupling in action (1) has attracted some attention and, in a recent paper [19], this possibility has been applied to distinct matter contents. Regarding the

latter, it was argued that a "natural choice" for the matter Lagrangian density for perfect fluids is $\mathcal{L}_m = p$, based on [12, 13], where p is the pressure. This specific choice implies the vanishing of the extra force. However, although $\mathcal{L}_m = p$ does indeed reproduce the perfect fluid equation of state, it is not unique: other choices include, for instance, $\mathcal{L}_m = -\rho$ [13, 14], where ρ is the energy density, or $\mathcal{L}_m = -na$, where n is the particle number density, and a is the physical free energy defined as $a = \rho/n - Ts$, with T being the fluid temperature and s the entropy per particle.

In this section, following [13, 15], the Lagrangian formulation of a perfect fluid in the context of GR is reviewed. The action is presented in terms of Lagrange multipliers along the Lagrange coordinates α^A in order to enforce specific constraints, and is given by

$$S_m = \int d^4x \left[-\sqrt{-g}\, \rho(n,s) + J^\mu \left(\varphi_{,\mu} + s\theta_{,\mu} + \beta_A \alpha^A_{,\mu} \right) \right].$$
(8)

Note that the action $S_m = S(g_{\mu\nu}, J^\mu, \varphi, \theta, s, \alpha^A, \beta_A)$ is a functional of the spacetime metric $g_{\mu\nu}$, the entropy per particle s, the Lagrangian coordinates α^A, and spacetime scalars denoted by φ, θ, and β_A, where the index A takes the values 1, 2, 3 (see [13] for details).

The vector density J^μ is interpreted as the flux vector of the particle number density, and defined as $J^\mu = \sqrt{-g}\, nU^\mu$. The particle number density is given by $n = |J|/\sqrt{-g}$, so that the energy density is a function $\rho = \rho(|J|/\sqrt{-g}, s)$. The scalar field φ is interpreted as a potential for the chemical free energy f, and is a Lagrange multiplier for $J^\mu_{,\mu}$, the particle number conservation. The scalar fields β_A are interpreted as the Lagrange multipliers for $\alpha^A_{,\mu} J^\mu = 0$, restricting the fluid 4−velocity to be directed along the flow lines of constant α^A.

The variation of the action with respect to J^μ, φ, θ, s, α^A and β_A, provides the equations of motion, which are not written here (we refer the reader to Ref. [15] for details). Varying the action with respect to the metric, and using the definition given by Eq. (3), provides the stress-energy tensor for a perfect fluid

$$T^{\mu\nu} = \rho U^\mu U^\nu + \left(n\frac{\partial\rho}{\partial n} - \rho \right) (g^{\mu\nu} + U^\mu U^\nu),$$
(9)

with the pressure defined as

$$p = n\frac{\partial\rho}{\partial n} - \rho.$$
(10)

This definition of pressure is in agreement with the First Law of Thermodynamics, $d\rho = \mu\, dn + nTds$. The latter shows that the equation of state can be specified by the energy density $\rho(n,s)$, written as a function of the number density and entropy per particle. The quantity $\mu = \partial\rho/\partial n = (\rho + p)/n$ is defined as the chemical potential, which is the energy gained by the system per particle injected into the fluid, maintaining a constant sample volume and entropy per particle s.

Taking into account the equations of motions and the definitions $J^\mu = \sqrt{-g}\, nU^\mu$ and $\mu = (\rho + p)/n$, the action Eq. (8) reduces to the on-shell Lagrangian density $\mathcal{L}_{m(1)} = p$, with the action given by [15]

$$S_m = \int d^4x \sqrt{-g}\, p,$$
(11)

which is the form considered in Ref. [12]. It was a Lagrangian density given by $\mathcal{L}_m = p$ that the authors of [19] use to obtain a vanishing extra-force due to the non-trivial coupling of matter to the scalar curvature R. For concreteness, replacing $\mathcal{L}_m = p$ in Eq. (7), one arrives at the general relativistic expression

$$f^\mu = \frac{h^{\mu\nu}\nabla_\nu p}{\rho + p}.$$
(12)

However, an on-shell degeneracy of the Lagrangian densities arises from adding up surface integrals to the action. For instance, consider the following surface integrals added to the action Eq. (8),

$$-\int d^4x(\varphi J^\mu)_{,\mu}, \quad -\int d^4x(\theta s J^\mu)_{,\mu},$$
$$-\int d^4x(J^\mu \beta_A \alpha^A)_{,\mu},$$

so that the resulting action takes the form

$$S = \int d^4x \left[-\sqrt{-g}\, \rho(n,s) - \varphi J^\mu_{,\mu} \right.$$
$$\left. -\, \theta(s J^\mu)_{,\mu} - \alpha^A(\beta_A J^\mu)_{,\mu} \right].$$
(13)

This action reproduces the equations of motion, and taking into account the latter, the action reduces to [15]

$$S_m = -\int d^4x\sqrt{-g}\, \rho,$$
(14)

i.e., the on-shell matter Lagrangian density takes the following form $\mathcal{L}_m = -\rho$. This choice is also considered for isentropic fluids, where the entropy per particle is constant $s = $ const. [13, 14]. For the latter, the First Law of Thermodynamics indicates that isentropic fluids are described by an equation of state of the form $a(n,T) = \rho(n)/n - sT$ [13] (see Ref. [20] for a bulk-brane discussion of this choice).

For this specific choice of $\mathcal{L}_{m(2)} = -\rho$ the extra force takes the following form:

$$f^\mu = \left(-\frac{\lambda F_2}{1 + \lambda f_2}\nabla_\nu R + \frac{1}{\rho + p}\nabla_\nu p \right) h^{\mu\nu}.$$
(15)

An interesting feature of Eq. (15) is that the term related to the specific curvature-matter coupling is independent of the energy-matter distribution.

The above discussion confirms that if one adopts a particular on-shell Lagrangian density as a suitable functional for describing a perfect fluid, then this leads to the issue of distinguishing between different predictions

for the extra force. It is therefore clear that no straight-forward conclusion may be extracted regarding the additional force imposed by the non-minimal coupling of curvature to matter, given the different available choices for the Lagrangian density. One could even doubt the validity of a conclusion that allows for different physical predictions arising from these apparently equivalent Lagrangian densities.

Despite the fact that the above Lagrangian densities $\mathcal{L}_{m(i)}$ are indeed obtainable from the original action, it turns out that they are not equivalent to the original Lagrangian density $\mathcal{L}_m$. Indeed, this equivalence demands that not only the equations of motion of the fields describing the perfect fluid remain invariant, but also that the gravitational field equations do not change. Indeed, the guiding principle behind the proposal first put forward in Ref. [8] is to allow for a non-minimal coupling between curvature and matter.

The modification of the perfect fluid action Eq. (8) should only affect the terms that show a minimal coupling between curvature and matter, i.e., those multiplied by $\sqrt{-g}$ [15]. Thus, the current density term, which is not coupled to curvature, should not be altered. Writing $\mathcal{L}_c = -\rho(n, s)$, $V_\mu \equiv \varphi_{,\mu} + s\theta_{,\mu} + \beta_A \alpha^A_{,\mu}$, for simplicity, the modified action reads

$$S'_m = \int d^4x \left[\sqrt{-g} \left[1 + \lambda f_2(R) \right] \mathcal{L}_c + J^\mu V_\mu + B^\mu_{;\mu} \right],$$
(16)

and one can see that only the non-minimal coupled term $\mathcal{L}_c$ appears in the field equations, as variations with respect to $g^{\mu\nu}$ of the remaining terms vanish:

$$F_1 R_{\mu\nu} - \frac{1}{2} f_1 g_{\mu\nu} - \nabla_\mu \nabla_\nu F_1 + g_{\mu\nu} \Box F_1 = (1 + \lambda f_2) T_{\mu\nu}$$
$$-2\lambda F_2 \mathcal{L}_c R_{\mu\nu} + 2\lambda (\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box) \mathcal{L}_c F_2.$$
(17)

Thus, quite logically, one finds that different predictions for non-geodesic motion are due to different forms of the gravitational field equations. Therefore, the equivalence between different on-shell Lagrangian densities $\mathcal{L}_{m(i)}$ and the original quantity $\mathcal{L}_m$ is broken, so that one can no longer freely choose between the available forms. For the same reason, the additional extra force is unique, and obtained by replacing $\mathcal{L}_c = -\rho$ into Eq. (7), yielding expression (15).

Indeed, in a recent paper [29], a generalization of the above approach is considered, by using a systematic method that is not tied up to a specific choice of matter Lagrangians. In particular, the propagation equations for pole-dipole particles for a gravity theory with a very general coupling between the curvature scalar and the matter fields is examined, and it is shown that, in general, the extra-force does not vanish.

IV. SCALAR-TENSOR REPRESENTATION

The connection between $f(R)$ theories of gravity and scalar-tensor models with a "physical" metric coupled to the scalar field is well known. In this section, one pursues the equivalence between the model described by Eq. (1) and an adequate scalar-tensor theory. In close analogy with the equivalence of standard $f(R)$ models [21], this equivalence allows for the calculation of the PPN parameters β and γ [22].

One may first approach this equivalence by introducing two auxiliary scalars ψ and ϕ [19], and considering the following action

$$S_1 = \int \left[\frac{1}{2} f_1(\phi) + [1 + \lambda f_2(\phi)] \mathcal{L}_m + \psi(R - \phi) \right] \sqrt{-g} \, d^4x.$$
(18)

Now, varying the action with respect to ψ gives $\phi = R$ and, consequently, action (1) is recovered. Varying the action with respect to ϕ, yields

$$\psi = \frac{1}{2} F_1 + \lambda F_2 \mathcal{L}_m.$$
(19)

Substituting this relationship back in (18), and assuming that at least one of the functions f_i is nonlinear in R, one arrives at the following modified action

$$S_1 = \int \left[\frac{f_1(\phi)}{2} + [1 + \lambda f_2(\phi)] \mathcal{L}_m \right.$$
$$\left. + \left[\frac{1}{2} F_1(\phi) + \lambda F_2(\phi) \mathcal{L}_m \right] (R - \phi) \right] \sqrt{-g} \, d^4x \quad (20)$$

where one still verifies the presence of the curvature-matter coupling. Note that this is not an ordinary scalar-tensor theory, due to the presence of the third and last terms. The former represents a scalar-matter coupling, and the latter a novel scalar-curvature-matter coupling. One may also use alternative field definitions to cast the action (18) into a Bran-Dicke theory with $\omega = 0$, i.e. no kinetic energy term for the scalar field, but with the addition of a R-matter coupling [19]. In conclusion, despite the fact that the introduction of the scalar fields helps in avoiding the presence of the nonlinear functions of R, the curvature-matter couplings are still present and, consequently, these actions cannot be cast into the form of a familiar scalar-tensor gravity [19].

However, one may instead pursue an equivalence with a theory with not just one, but two scalar fields [16]. This is physically well motivated, since the non-minimal coupling of matter and geometry embodied in Eq. (1) gives rise to an extra degree of freedom (notice that the case of a minimal coupling $f_2 = 0$ yields $\psi = F_1(\phi)/2$, so that this degree of freedom is lost). Indeed, action of Eq. (18) may be rewritten as a Jordan-Brans-Dicke theory with a suitable potential,

$$S_1 = \int \left[\psi R - V(\phi, \psi) + [1 + \lambda f_2(\phi)] \mathcal{L}_m \right] \sqrt{-g} \, d^4x,$$
(21)

with $V(\phi, \psi) = \phi\psi - f_1(\phi)/2$.

Variation of this action yields the field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G\frac{1 + \lambda f_2(\phi)}{\psi}T_{\mu\nu} \qquad (22)$$

$$-\frac{1}{2}g_{\mu\nu}\frac{V(\phi,\psi)}{\psi} + \frac{1}{\psi}\left(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box\right)\psi$$

which, after the substitutions $\phi = R$ and $\psi = F_1/2 + \lambda F_2\mathcal{L}_m$, collapses back to Eqs. (2). Likewise, the Bianchi identities yield the generalized covariant conservation equation

$$\nabla^\mu T_{\mu\nu} = \frac{1}{1 + \lambda f_2}\left[(\phi - R)\nabla_\nu\psi + \qquad (23)\right.$$

$$\left.\left[\left(\psi - \frac{1}{2}F_1\right)g_{\mu\nu} - \lambda F_2 T_{\mu\nu}\right]\nabla^\mu\phi\right],$$

also equivalent to Eq. (4).

Through a conformal transformation $g_{\mu\nu} \to g_{\mu\nu}^* = \psi g_{\mu\nu}$ (see *e.g.* [23]), the scalar curvature can decouple from the scalar fields, so that the action is written in the so-called Einstein frame). A further redefinition of the scalar fields,

$$\varphi^1 = \frac{\sqrt{3}}{2}\log\psi \quad , \quad \varphi^2 = \phi, \qquad (24)$$

allows the theory to be written canonically, that is,

$$S_1 = \int\left[R^* - 2g^{*\mu\nu}\sigma_{ij}\varphi_{,\mu}^i\varphi_{,\nu}^j \qquad (25)\right.$$

$$\left.-4U(\varphi^1,\varphi^2) + \left[1 + \lambda f_2(\varphi^2)\right]\mathcal{L}_m^*\right]\sqrt{-g^*}\,d^4x,$$

with $\mathcal{L}_m^* = \mathcal{L}_m/\psi^2$, the redefined potential

$$U(\varphi^1,\varphi^2) = \frac{1}{4}\exp\left(-\frac{2\sqrt{3}}{3}\varphi^1\right)\times \qquad (26)$$

$$\left[\varphi^2 - \frac{1}{2}f_1(\varphi^2)\exp\left(-\frac{2\sqrt{3}}{3}\varphi^1\right)\right],$$

and the metric in the field space (φ^1,φ^2),

$$\sigma_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad (27)$$

which, after a suitable addition of an anti-symmetric part, will be used to raise and lower Latin indexes.

Variation of action Eq. (25) with respect to the metric $g_{\mu\nu}^*$ yields the field equations

$$R_{\mu\nu}^* - \frac{1}{2}g_{\mu\nu}^*R^* = 8\pi G\left(1 + \lambda f_2\right)T_{\mu\nu}^* + \qquad (28)$$

$$\sigma_{ij}\left(2\varphi_{,\mu}^i\varphi_{,\nu}^j - g_{\mu\nu}^*g^{*\alpha\beta}\varphi_{,\alpha}^i\varphi_{,\beta}^j\right) - 2g_{\mu\nu}^*U,$$

while variation with respect to φ^i gives the Euler-Lagrange equations for each field:

$$\Box^*\varphi^i = B^i + 4\pi G\left[\alpha^i\left(1 + \lambda f_2\right)T^* - \lambda\sigma^{i2}F_2\mathcal{L}^*\right] \quad (29)$$

where one defines $B_i = \partial U/\partial\varphi^i$ and

$$\alpha_i = -\frac{1}{2}\frac{\partial\log\psi}{\partial\varphi^i} \quad \to \quad \alpha_1 = -\frac{\sqrt{3}}{3} \quad , \quad \alpha_2 = 0, \quad (30)$$

Eqs. (28), together with the Bianchi identities, result in the generalized conservation law

$$\nabla^{*\mu}T_{\mu\nu}^* = \frac{\sqrt{3}}{3}T^*\nabla_\nu^*\varphi^1 + \frac{\lambda F_2}{1 + \lambda f_2}\left(g_{\mu\nu}^*\mathcal{L}^* - T_{\mu\nu}^*\right)\nabla^{*\mu}\varphi^2. \qquad (31)$$

From current bounds on the Equivalence Principle, it is reasonable to assume that the effect of the non-minimum coupling of curvature to matter is weak, $\lambda f_2 \ll 1$. Substituting this into (31) one gets, at zeroth-order in λ,

$$\nabla^{*\mu}T_{\mu\nu}^* \simeq -\alpha_j T^*\varphi_{,\nu}^j, \qquad (32)$$

so that one may disregard the $f_2(\varphi^2)$ factor in the action (25) and consider only through the coupling present in T^* (stemming from the definition of $\mathcal{L}_m^*$) and the derivative of φ^1 (since $\varphi^1 \propto \log\psi$ and $\psi = F_1 + F_2\mathcal{L}$).

If both scalar fields are light, leading to long range interactions, one may calculate the PPN parameters β and γ [22], given by

$$\beta - 1 = \frac{1}{2}\left[\frac{\alpha^i\alpha^j\alpha_{j,i}}{(1 + \alpha^2)^2}\right]_0 \quad , \quad \gamma - 1 = -2\left[\frac{\alpha^2}{1 + \alpha^2}\right]_0, \qquad (33)$$

where $\alpha_{j,i} = \partial\alpha_j/\partial\varphi^i$ and $\alpha^2 = \alpha_i\alpha^i = \sigma^{ij}\alpha_i\alpha_j$; the subscript $_0$ refers to the asymptotic value of the related quantities, which is connected to the cosmological values of the curvature and matter Lagrangian density. From the values found in Eq. (30), one concludes that $\beta = \gamma = 1$, as obtained in GR. However, it should be expected that small deviations of order $O(\lambda)$ arise when one considers the full impact of Eq. (31).

Furthermore, it should be empathized that the added degree of freedom embodied in the non-minimal $f_2 \neq 0$ coupling is paramount in obtaining values for the PPN parameters β and γ within the current experimental bounds (or, conversely, allowing for future constraints of the magnitude of λ and the form of f_2); indeed, in the case where only the curvature term is non-trivial, $f_1 \neq R$ and $f_2 = 0$, one degree of freedom is lost and the parameter $\alpha \neq 0$ defined in Eq. (30) is no longer a vector, but a scalar quantity: as a result, $\alpha^2 \neq 0$ and one gets $\gamma = 1/2$. In the discussed model, the vector α_i has $\alpha^2 = 0$, thus solving this pathology (see [16] and references therein for a thorough discussion).

Finally, notice that these results should be independent of the particular scheme chosen for the equivalence between the original model and a scalar-tensor theory; this may be clearly seen by opting for a more "natural"

choice for the two scalar fields (in the Jordan frame), such that $\phi = R$ and $\psi = \mathcal{L}$. Although more physically motivated, this choice of fields is less pedagogical and mathematically more taxing [16].

V. IMPLICATIONS FOR STELLAR EQUILIBRIUM

In this section, one studies the impact of the non-minimally coupled gravity model embodied in action Eq. (1) in what may be viewed as its natural proving ground: regions where curvature effects may be high enough, to evidence some deviation from GR, although moderate enough so these are still perturbative – a star [17] (see also [24] for other physical examples of the adopted methodology). As will be shown, the purpose of this exercise is to calculate deviations to the central temperature of the Sun (known with an accuracy of 6%), due to the perturbative effect of the non-minimal coupling of geometry to matter.

Clearly, a full treatment of the equations of motion (2) is exceedingly demanding, unless a specific form for $f_1(R)$ and $f_2(R)$ is considered. Furthermore, since one is mainly interested in the ascertaining the effects of the non-minimal coupling within a high curvature and pressure medium, the modifications due to the pure curvature term f_1 should be overwhelmed by the effect of f_2; under such circumstances, one may discard the former term, as thus take the trivial $f_1 = R$ case. A thorough discussion on the validity of this approximation with regard to representative, physically viable candidates for the function $f_1(R)$ is found in Ref. [17].

One now deals with the particular form of the coupling function f_2. One considers the simplest form, which might arise from the first order expansion of a more general function in the weak field environ of the Sun, $f_2 = R$ (this implies that $[\lambda] = M^{-2}$). Also, one assumes that stellar matter is described by an ideal fluid characterized by a Lagrangian density $\mathcal{L}_m = p$, [12, 13]. Adopting $f_1 = f_2 = R$, the field equations become

$$(1 + 2\lambda p)\, R_{\mu\nu} - \frac{1}{2} R\, (g_{\mu\nu} + 2\lambda T_{\mu\nu}) = \qquad (34)$$
$$2\lambda(\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box)p + \frac{1}{2} T_{\mu\nu} \quad ,$$

Notice that both λp and $\lambda \rho$ are dimensionless quantities: the perturbative condition $\lambda f_2 \ll 1$ translates to $\lambda p \ll 1$ and $\lambda \rho \ll 1$.

Taking the trace of the above equation yields

$$R = \frac{3p - \rho + 6\lambda \Box p}{2\left[1 + \lambda(\rho - 5p)\right]} \quad , \qquad (35)$$

inserting $T = T_\mu^\mu = \rho - 3p$. Substituting this into Eq. (34) and keeping only first order terms in λ, one obtains

$$2[1 + \lambda(\rho - 3p)]R_{\mu\nu} = \qquad (36)$$
$$(3p - \rho)g_{\mu\nu} + 2(1 - 2\lambda p)T_{\mu\nu} + 2\lambda(4\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)p \quad ,$$

Since temporal variations are assumed to occur at the cosmological scale H_0^{-1}, and are thus negligible at an astrophysical time scale, one considers an ideal, spherically symmetric system, with a line element derived from the Birkhoff metric (in its anisotropic form)

$$ds^2 = e^{\nu(r)}dt^2 - \left(e^{\sigma(r)}dr^2 + d\Omega^2\right) \quad , \qquad (37)$$

with $d\Omega = r^2(d\theta^2 + sin^2\theta\ d\phi^2)$. Following the usual treatment, one defines the effective mass m_e through $e^{-\sigma} = 1 - 2Gm_e/r$ which, replacing in Eq. (36) yields, to first order in λ,

$$m_e' \approx 4\pi r^2 \rho \left[1 + 2\lambda\left(p - \frac{\rho}{2} - \frac{3}{2}\frac{p^2}{\rho}\right)\right] + \qquad (38)$$
$$\frac{\lambda r^2}{4G}\left(5e^{-\nu}\nabla_0\nabla_0 + 3e^{-\sigma}\nabla_r\nabla_r + 2\frac{\nabla_\theta\nabla_\theta}{r^2}\right)p \quad ,$$

which clearly shows the perturbation to the gravitational mass, defined by $m_g' = 4\pi r^2 \rho$ (in here, the prime denotes differentiation with respect to r).

Taking the Newtonian limit

$$r \gg 2Gm_e(r)\ ,\ \rho(r) \gg p(r)\ ,\ m_e(r) \gg 4\pi p(r)r^3\ , \quad (39)$$

and going through a few algebraic steps (depicted in [17]), one eventually obtains the non-relativistic hydrostatic equilibrium equation

$$p' + \frac{Gm_e\rho}{r^2} = 2\lambda\left[\left(\left[\frac{5}{8}p'' - 4\pi Gp\rho\right]r - \frac{p'}{4}\right)\rho + p\rho'\right] \quad . \qquad (40)$$

where the perturbation introduced by the non-minimal coupling is clearly visible.

In order to scrutinize the profile of pressure and density inside the Sun, one requires a suitable equation of state. Instead of pursuing a realistic representation of the various layers of the solar structure, one resorts to a very simplistic assumption, the so-called polytropic equation of state. This is commonly given by $p = K\rho^{(n+1)/n}$, where K is the polytropic constant, ρ is the mass density and n is the polytropic index. A polytropic equation of state with $n = 3$ was used by Eddington in his first solar model, and will be adopted here due.

Given this equation of state, one may write $\rho = \rho_c\theta^n(\xi)$ and $p = p_c\theta^{n+1}(\xi)$, with $\xi = r/r_0$ a dimensionless variable and $r_0^2 \equiv (n+1)p_c/4\pi G\rho_c^2$; $\rho_c = 1.622 \times 10^5$ kg/m^3 is the central density, and $p_c = 2.48 \times 10^{16}$ Pa is the central pressure. One obtains the perturbed Lane-Emden equation for the function $\theta(\xi)$:

$$\frac{1}{\xi^2}\left[\xi^2\theta'\left(1 + A_c\theta^n \times \qquad (41)\right.\right.$$
$$\left.\left[\left[\frac{5}{8}\left(\theta'' + n\frac{\theta'^2}{\theta}\right) - N_c\theta^{n+1}\right]\frac{\xi}{\theta'} + \frac{3n-1}{4(n+1)}\right]\right)\right]' =$$
$$-\theta^n\left[1 + A_c\left(\frac{3}{8}\left[\theta'' + n\frac{\theta'^2}{\theta}\right] + \frac{\theta'}{4\xi} - \frac{\theta^n}{2}\right)\right] \quad ,$$
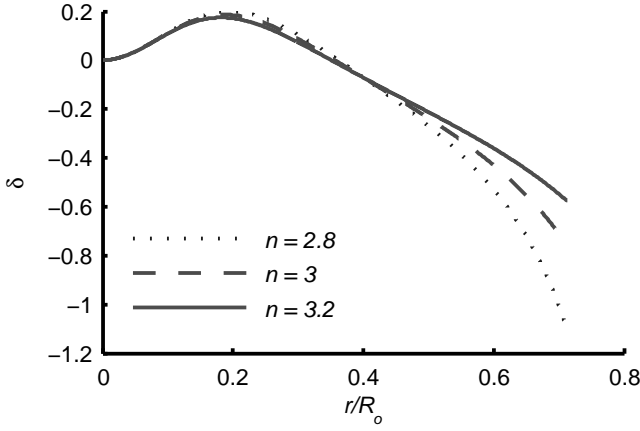
FIG. 1: Relative perturbation δ for $2.8 \leq n \leq 3.2$.

FIG. 2: Relative deviation of the central temperature $T_c/T_{c0} - 1$, with contour lines of step 0.1%.

where the prime now denotes derivation with respect to the dimensionless radial coordinate ξ, and one defines the dimensionless parameters $A_c \equiv \lambda \rho_c$ and $N_c \equiv p_c/\rho_c = 1.7 \times 10^{-6}$, for convenience. Clearly, setting $A_c = 0$ one recovers the unperturbed Lane-Emden equation [25].

Notice that the perturbed Lane-Emden equation is a third-degree differential equation; its numerical resolution is computationally intensive and displays some complex behaviour; conveniently, the assumed perturbative regime prompts for the expansion of the function $\theta(\xi) = \theta_0(1 + A_c\delta)$ around the unperturbed solution $\theta_0(\xi)$. Inserting this into Eq. (41) and expanding to first-order in A_c, one obtains

$$\delta'' + 2\left(\frac{\theta_0'}{\theta_0} + \frac{1}{\xi}\right)\delta' + (n-1)\theta_0^{n-1}\delta = \frac{5n}{2}\xi\theta_0^{2n-2}\theta_0' \quad (42)$$

$$+(2n+1)N_c\xi\theta_0^{2n-1}\theta_0' + \frac{9n+5}{4(n+1)}\theta_0^{2n-1} + 3N_c\theta_0^{2n}$$

$$-\frac{5n(n-1)}{8}\xi\theta_0^{n-3}\theta_0'^3 + \frac{n(3n+7)}{4(n+1)}\theta_0^{n-2}\theta_0'^2 + \frac{1}{2}\frac{\theta_0^{n-1}\theta_0'}{\xi} \quad .$$

supplemented by the initial conditions $\delta(0) = \delta'(0) = 0$. Notice that the choice for the perturbative expansion leads to a solution δ independent from the parameter A_c.

After dealing with the issue of exterior matching conditions and bypassing a troublesome divergence of δ near the boundary of the star [17], one may obtain the numerical solution for Eq. (42) for a polytropic index in the vicinity of $n = 3$, as depicted in Fig. 1.

Finally, one turns to the issue of calculating one of the observables under scrutiny, that is, the central temperature of the Sun. The polytropic equation of state indicates that $\rho \propto T^{n+1}$, which yields

$$1 - \left(\frac{T_{c0}}{T_c}\right)^{n+1} = \quad (43)$$

$$\frac{A_c}{\xi_r^2\theta_{0r}'}\int_0^{\xi_r}\xi^2\theta_0^n\left[n\delta + \frac{3n}{8}\frac{\theta_0'^2}{\theta_0} - \frac{\theta_0'}{2\xi} - \frac{7}{8}\theta_0^n\right]\,d\xi \quad .$$

where $\xi_r = R_r/r_0$ and $R_r = 0.713R_\odot$ marks the onset of the convection zone (where the chosen equation of state

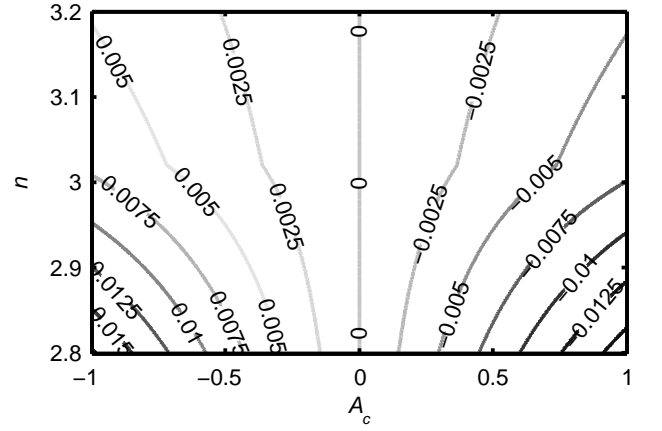fails) and T_{c0} is the central temperature derived from the $A_c = 0$ unperturbed scenario. One may derive a parameter plot in the (n, A_c) parameter space, shown in Fig. 2. As can be seen, no relative deviation of the central temperature occurs above the experimentally determined level of 6%. However, since the values found are of the order of 1%, one may hope that any future refinement of the experimental error of T_c could yield a direct bound on the parameter A_c. Furthermore, the perturbative condition $\lambda \ll \kappa\rho_c$ is confirmed (reintroducing the factor κ, for clarity), which translates to $|\lambda| \ll 4.24 \times 10^{33}$ eV^{-2}.

VI. MODELS WITH ARBITRARY COUPLINGS BETWEEN MATTER AND GEOMETRY

The discussed gravity models with linear coupling between matter and geometry, given by Eq. (1), can be further generalized by assuming that the supplementary coupling between matter and geometry takes place via an arbitrary function of the matter Lagrangian $\mathcal{L}_m$, so that the action is given by [18]

$$S = \int\left\{\frac{1}{2}f_1(R) + \mathcal{G}\left(\mathcal{L}_m\right)\left[1 + \lambda f_2\left(R\right)\right]\right\}\sqrt{-g}d^4x,$$
$$(44)$$

where $\mathcal{G}\left(\mathcal{L}_m\right)$ is an arbitrary function of the matter Lagrangian density $\mathcal{L}_m$. The action given by Eq. (44) represents the most general extension of the Einstein-Hilbert action for GR, $S = \int\left[R/2 + \mathcal{L}_m\right]\sqrt{-g}d^4x$. For $f_1(R) = R$, $f_2(R) = 0$ and $\mathcal{G}\left(\mathcal{L}_m\right) = \mathcal{L}_m$, one recovers GR. With $f_2(R) = 0$ and $\mathcal{G}\left(\mathcal{L}_m\right) = \mathcal{L}_m$ one obtains the $f(R)$ generalized gravity models. The case $\mathcal{G}\left(\mathcal{L}_m\right) = \mathcal{L}_m$ corresponds to the linear coupling between matter and geometry, given by Eq. (1). The only requirement for f_i, $i = 1, 2$ and $\mathcal{G}$ is that they are analytical functions of the Ricci scalar R and $\mathcal{L}_m$, respectively – that is, they can be expressed as a Taylor series expansion about any point.

The field equations corresponding to action (44) are

$$F_1(R)R_{\mu\nu} - \frac{1}{2}f_1(R)g_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)F_1(R) =$$
$$-2\lambda\mathcal{G}(\mathcal{L}_m)F_2(R)R_{\mu\nu}$$
$$-2\lambda(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)\mathcal{G}(\mathcal{L}_m)F_2(R)$$
$$-[1 + \lambda f_2(R)][K(\mathcal{L}_m)\mathcal{L}_m - \mathcal{G}(\mathcal{L}_m)]g_{\mu\nu}$$
$$-[1 + \lambda f_2(R)]K(\mathcal{L}_m)T_{\mu\nu}, \tag{45}$$

where $F_i(R) = df_i(R)/dR$, $i = 1, 2$, and $K(\mathcal{L}_m) = d\mathcal{G}(\mathcal{L}_m)/d\mathcal{L}_m$, respectively.

By taking the covariant divergence of Eq. (45), with the use of the mathematical identity $\nabla^\mu[a'(R)R_{\mu\nu} - a(R)g_{\mu\nu}/2 + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)a(R)] = 0$ [26], where $a(R)$ is an arbitrary function of the Ricci scalar and $a'(R) = da/dR$, we obtain

$$\nabla^\mu T_{\mu\nu} = \nabla^\mu\ln\{[1 + \lambda f_2(R)]K(\mathcal{L}_m)\}(\mathcal{L}_m g_{\mu\nu} - T_{\mu\nu})$$
$$= 2\nabla^\mu\ln\{[1 + \lambda f_2(R)]K(\mathcal{L}_m)\}\frac{\partial\mathcal{L}_m}{\partial g^{\mu\nu}}. \tag{46}$$

For $\mathcal{G}(\mathcal{L}_m) = \mathcal{L}_m$, one recovers the equation of motion of massive test particles in the linear theory, Eq. (7). As a specific model of generalized gravity models with arbitrary matter-geometry coupling, one considers the case in which the matter Lagrangian density is an arbitrary function of the energy density of the matter ρ only, so that $\mathcal{L}_m = \mathcal{L}_m(\rho)$. One assumes that during the hydrodynamic evolution the energy density current is conserved, $\nabla_\nu(\rho U^\nu) = 0$. Then, the energy-momentum tensor of matter is given by

$$T^{\mu\nu} = \rho\frac{d\mathcal{L}_m}{d\rho}U^\mu U^\nu + \left(\mathcal{L}_m - \rho\frac{d\mathcal{L}_m}{d\rho}\right)g^{\mu\nu}, \tag{47}$$

where we have used the relation $\delta\rho = (1/2)\rho(g_{\mu\nu} - U_\mu U_\nu)\delta g^{\mu\nu}$, a direct consequence of the conservation of the energy density current.

The energy-momentum tensor given by Eq. (47) can be written in a form similar to the perfect fluid case if one assumes that the thermodynamic pressure p obeys a barotropic equation of state, $p = p(\rho)$. In this case the matter Lagrangian density and the energy-momentum tensor can be written as

$$\mathcal{L}_m(\rho) = \rho[1 + \Pi(\rho)] = \rho\left(1 + \int_0^p\frac{dp}{\rho}\right) - p(\rho), \tag{48}$$

and

$$T^{\mu\nu} = \{\rho[1 + \Pi(\rho)] + p(\rho)\}U^\mu U^\nu + p(\rho)g^{\mu\nu}, \tag{49}$$

respectively, where

$$\Pi(\rho) = \int_0^p\frac{dp}{\rho} - \frac{p(\rho)}{\rho}. \tag{50}$$

Physically, $\Pi(\rho)$ can be interpreted as the elastic (deformation) potential energy of the body, and therefore

Eq. (49) corresponds to the energy-momentum tensor of a compressible elastic isotropic system. From Eq. (46), one obtains the equation of motion of a test particle in the modified gravity model with the matter Lagrangian an arbitrary function of the energy density of matter as Eq. (6), where the extra force is now given by

$$f^\mu = \nabla_\nu\ln\left\{[1 + \lambda f_2(R)]K[\mathcal{L}_m(\rho)]\frac{d\mathcal{L}_m(\rho)}{d\rho}\right\}h^{\mu\nu}. \tag{51}$$

It is easy to see that the extra-force f^μ, generated due to the presence of the coupling between matter and geometry, is perpendicular to the four-velocity, $f^\mu U_\mu = 0$. The equation of motion, Eq. (6), can be obtained from the variational principle

$$\delta S_p = \delta\int\mathcal{L}_p ds = \delta\int\sqrt{Q}\sqrt{g_{\mu\nu}U^\mu U^\nu}ds = 0, \tag{52}$$

where S_p and $\mathcal{L}_p = \sqrt{Q}\sqrt{g_{\mu\nu}U^\mu U^\nu}$ are the action and Lagrangian density, respectively, and

$$\sqrt{Q} = [1 + \lambda f_2(R)]K[\mathcal{L}_m(\rho)]\frac{d\mathcal{L}_m(\rho)}{d\rho}. \tag{53}$$

The variational principle Eq. (52) can be used to study the Newtonian limit of the model. In the weak gravitational field limit, $ds \approx \sqrt{1 + 2\phi - \vec{v}^2}dt \approx (1 + \phi - \vec{v}^2/2)dt$, where ϕ is the Newtonian potential and $\vec{v}$ is the usual tridimensional velocity of the particle. By representing the function $\sqrt{Q}$ as

$$\sqrt{Q} = [1 + \lambda f_2(R)]K[\mathcal{L}_m(\rho)]\frac{d\mathcal{L}_m(\rho)}{d\rho}$$
$$= 1 + \Phi\left(R, \mathcal{L}_m(\rho), \frac{d\mathcal{L}_m(\rho)}{d\rho}\right), \tag{54}$$

where $|\Phi| << 1$, the equation of motion of a test particle can be obtained from the variational principle

$$\delta\int\left[\Phi\left(R, \mathcal{L}_m(\rho), \frac{d\mathcal{L}_m(\rho)}{d\rho}\right) + \phi - \frac{\vec{v}^2}{2}\right]dt = 0, \tag{55}$$

and is given by

$$\vec{a} = -\nabla\phi - \nabla\Phi = \vec{a}_N + \vec{a}_E, \tag{56}$$

where $\vec{a}_N = -\nabla\phi$ is the usual Newtonian gravitational acceleration and $\vec{a}_E = -\nabla\Phi$ a supplementary effect induced by the coupling between matter and geometry.

An estimative of the effect of the extra-force generated by the coupling between matter and geometry on the orbital parameters of planetary motion around the Sun can be obtained by using the properties of the Runge-Lenz vector, defined as $\vec{A} = \vec{v}\times\vec{L} - \alpha\vec{e}_r$, where $\vec{v}$ is the velocity relative to the Sun, with mass $M_\odot$, of a planet of mass m, $\vec{r} = r\vec{e}_r$ is the two-body position vector, $\vec{p} = \mu\vec{v}$ is the relative momentum, $\mu = mM_\odot/(m + M_\odot)$ is the reduced mass, $\vec{L} = \vec{r}\times\vec{p} = \mu r^2\dot{\theta}\vec{k}$ is the angular momentum,

and $\alpha = GmM_\odot$ [27]. For an elliptical orbit of eccentricity e, major semi-axis a, and period T, the equation of the orbit is given by $\left(L^2/\mu\alpha\right)r^{-1} = 1 + e\cos\theta$. The Runge-Lenz vector and its derivative can be expressed as

$$\vec{A} = \left(\frac{\vec{L}^2}{\mu r} - \alpha\right)\vec{e}_r - \dot{r}L\vec{e}_\theta,\qquad(57)$$

and

$$\frac{d\vec{A}}{d\theta} = r^2\left[\frac{dV(r)}{dr} - \frac{\alpha}{r^2}\right]\vec{e}_\theta,\qquad(58)$$

respectively, where $V(r)$ is the potential of the central force [27]. The potential term consists of the Post-Newtonian potential,

$$V_{PN}(r) = -\frac{\alpha}{r} - \frac{3\alpha^2}{mr^2},\qquad(59)$$

plus the contribution from the general coupling between matter and geometry. Thus, one has

$$\frac{d\vec{A}}{d\theta} = r^2\left[\frac{6\alpha^2}{mr^3} + m\vec{a}_E(r)\right]\vec{e}_\theta,\qquad(60)$$

where it is also assumed that $\mu \approx m$. The change in direction $\Delta\phi$ of the perihelion for a variation of θ of 2π is obtained as

$$\Delta\phi = \frac{1}{\alpha e}\int_0^{2\pi}\left|\dot{\vec{L}}\times d\frac{\vec{A}}{d\theta}\right|d\theta,\qquad(61)$$

and is given by

$$\Delta\phi = 24\pi^3\left(\frac{a}{T}\right)^2\frac{1}{1-e^2} + \frac{L}{8\pi^3 me}\frac{\left(1-e^2\right)^{3/2}}{\left(a/T\right)^3}\times$$
$$\int_0^{2\pi}\frac{a_E\left[L^2\left(1+e\cos\theta\right)^{-1}/m\alpha\right]}{\left(1+e\cos\theta\right)^2}\cos\theta d\theta,\qquad(62)$$

where the relation $\alpha/L = 2\pi\left(a/T\right)/\sqrt{1-e^2}$ is used. The first term of this equation corresponds to the GR prediction for the precession of the perihelion of planets, while the second gives the contribution to the perihelion precession due to the presence of the new coupling between matter and geometry.

As an example of the application of Eq. (62), one considers the case for which the extra-force a_E may be considered constant — an approximation that might be valid for small regions of the space-time. Thus, through Eq. (62), one obtains the perihelion precession

$$\Delta\phi = \frac{6\pi GM_\odot}{a\left(1-e^2\right)} + \frac{2\pi a^2\sqrt{1-e^2}}{GM_\odot}a_E,\qquad(63)$$

resorting to Kepler's third law, $T^2 = 4\pi^2 a^3/GM_\odot$.

For Mercury, $a = 57.91\times10^9$ m and $e = 0.205615$, respectively, while $M_\odot = 1.989\times10^{30}$ kg: the first term in Eq. (63) gives the GR value for the precession angle, $(\Delta\phi)_{GR} = 42.962$ arcsec per century, while the observed value is $(\Delta\phi)_{obs} = 43.11\pm0.21$ arcsec per century [28]. Therefore, the difference $(\Delta\phi)_E = (\Delta\phi)_{obs} - (\Delta\phi)_{GR} = 0.17$ arcsec per century can be attributed to other physical effects. Hence, the observational constraints requires that the value of the constant extra acceleration a_E must satisfy the condition

$$a_E \leq 1.28\times10^{-11}\text{ m/s}^2.\qquad(64)$$

This value of a_E, obtained from the solar system observations, is somewhat smaller than the value of the extra-acceleration $a_0 \approx 10^{-10}$ m/s^2, necessary to account for the Pioneer anomaly [8]. However, it does not rule out the possibility of the presence of some extra gravitational effects acting at both solar system and galactic scale, since the assumption of a constant extra-force may not be correct on large astronomical scales.

VII. CONCLUSIONS AND OUTLOOK

In this contribution we have discussed a wide range of implications of the gravity model action, Eq. (1), whose main feature is the non-minimal coupling between curvature and the Lagrangian density of matter (or a function of it, in Section VI). This exhibits an extra force with respect to the GR motion, as well as the non-conservation of the matter energy-momentum tensor. The prevalence of these features for different choices for the matter Lagrangian density was discussed in Section III. In Section IV, the specific features of the associated scalar-tensor theory were discussed — and it was shown that the model is consistent with the observational values of the PPN parameters, namely $\beta = \gamma = 1$, to zeroth-order in λ. In Section V, we consider the impact of the novel coupling on the issue of stellar equilibrium. It is shown that, for the simplest model of the Sun, the effect of the new coupling on the central temperature is smaller than 1 %, which is consistent with the uncertainty of current estimates. Finally, in Section VI, a general function of the matter Lagrangian density has been introduced, and the value of the resulting extra force obtained, $a_E \leq 10^{-11}$ m/s^2.

Of course, further work is still required in order to quantify the violation of the Equivalence Principle introduced by the model under realistic physical conditions. A low bound for the coupling λ, would justify the results discussed in this work, which are first order in λ. Implications of the discussed model in what concerns the issue of singularities are still to be addressed, as well as the impact that the new coupling term might have on the early Universe cosmology.

We would like to close this contribution with our best wishes to our colleague Sergei Odintsov, on the occasion of his 50th birthday.

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